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THE MOTION OF A VARIABLE BODY IN AN IDEAL FLUID†

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The dynamics of a deformable body in an unbounded volume of an ideal fluid, which performs irrotational motion and is at rest at infinity, is investigated. It is assumed that a change in the geometry of the masses and shape of the body occurs due to the action of internal forces and that the displacements of the particles of the body are known functions of time in a certain moving frame of reference. The equations of motion of the moving trihedron are represented in the form of Kirchhoff's equations. The conservation laws when there are no external forces are indicated. Using these laws, the equations of motion are reduced to a non-autonomic system of first-order differential equations in the group of displacements of the configurational space. In the case of plane-parallel motion of the body, these equations are explicitly integrated in quadratures. A special case, when the boundary of the body does not change, is considered. It is established that, in the case of non-equal added masses, due to the change in the geometry of the body masses, the body can move from any position into any other position.

1. GENERALIZED LIOUVILLE EQUATIONS

In 1858, Liouville [1] considered the general problem of the rotation of a deformable body around a moving point, where the geometry of the masses of the body only changes under the action of internal forces. He obtained the generalized dynamic Euler equations relative to moving axes which, at each instant of time, coincide with the principal axes of inertia of the body. Various aspects of Liouville's equations are discussed in Routh's book [2]. The Liouville problem can be extended by considering the three-dimensional motions of a variable body. It is found that the generalized Liouville equations admit of a unique representation in the form of Kirchhoff's equations, which is convenient from the point of view of the more general problem of the motion of a variable body in a fluid.

Thus, we refer the motion of a system of point masses with masses m_k to two Cartesian frames of reference: a fixed (inertial) frame $Oxyz$ and a moving reference frame $O_1\xi\eta\zeta$ (Fig. 1). We emphasize that (unlike the well-known approach in [1]) it is optional whether the point O_1 coincides with the centre of mass of the system of points and, in the general case, the ξ, η and ζ axes are not the axes of inertia of the system. Suppose r_k is the radius vector of the k-th point with respect to the fixed reference frame F_k and Φ_k are external and internal forces acting on the k-th point respectively and r_0 is the radius vector of the point O_1 with respect to point O. We put $r_k = r_0 + \rho_k$.

We recall that any vector function of time $f(t)$ can also be considered in the moving reference frame. We shall denote its derivative with respect to the moving frame (the relative velocity) by a dot. The absolute and relative velocities are related by Euler's formula

$$df/dt = \dot{f} + [\omega, f]$$

where ω is the angular velocity of the moving reference frame.

The kinetic energy of the system of points is as follows:

$$T = \frac{1}{2}\sum m_k\left(\frac{dr_k}{dt}, \frac{dr_k}{dt}\right), \quad \frac{dr_k}{dt} = v + \dot{\rho} + [\omega, \rho_k] \tag{1.1}$$

where $v = dr_0/dt$ is the velocity of the origin of the moving reference frame. It is henceforth assumed that the change in the geometry of the masses of the system of points under the action of internal forces is known. In other words, it is assumed that ρ_k are known functions of time. Note that other formulations of the problem are also possible. For example, Zeiliger and Chetayev (see [3]) studied the rotation of a body around a fixed point taking account of its radiative expansion.

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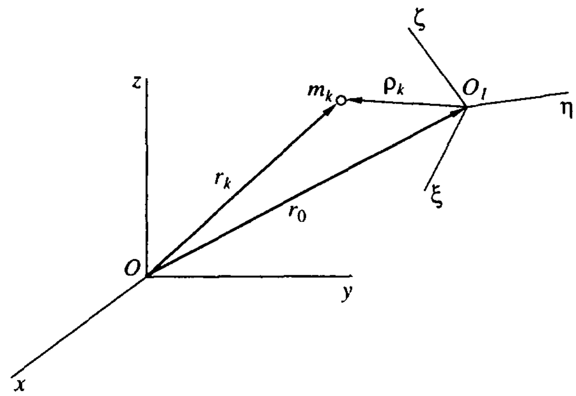

Fig. 1

It follows from relations (1.1) that

$$\frac{\partial T}{\partial v} = mv + \sum m_k \dot{\rho}_k + \sum m_k [\omega, \rho_k], \quad m = \sum m_k$$

which is the momentum P of the variable body. By the theorem and the change in momentum, $dP/dt = F = \Sigma F_k$. In the moving reference frame, this equation takes the form

$$\left(\frac{\partial T}{\partial v}\right)^{\cdot} + \left[\omega, \frac{\partial T}{\partial v}\right] = F \tag{1.2}$$

We now make use of a theorem on the change in the angular momentum about the point O

$$\frac{d}{dt} \sum m_k \left[r_k, \frac{dr_k}{dt}\right] = \sum [r_k, F_k + \Phi_k] \tag{1.3}$$

Taking account of the notation adopted, the angular momentum of the body is equal to

$$\frac{\partial T}{\partial \omega} + [r_0, P] \tag{1.4}$$

where

$$\frac{\partial T}{\partial \omega} = \sum m_k [\rho_k, [\omega, \rho_k]] + \sum m_k [\rho_k, \dot{\rho}_k] + \left[\sum m_k \rho_k, v\right] \tag{1.5}$$

The first and second terms in this formula have the meaning of the angular momentum of the system about the point O_1 for the reference-frame and relative motions of the body respectively. The third term vanishes if the point O_1 coincides with the centre of mass of the variable body.

Since the moment of the internal forces is equal to zero, from relations (1.3) and (1.4) we have the equation

$$\frac{d}{dt}\frac{\partial T}{\partial \omega} + \left[v, \frac{\partial T}{\partial v}\right] = \sum [r_k, F_k] - [r_0, F]$$

In the moving axes, it takes the following form

$$\left(\frac{\partial T}{\partial v}\right)^{\cdot} + \left[\omega, \frac{\partial T}{\partial \omega}\right] + \left[v, \frac{\partial T}{\partial v}\right] = M \tag{1.6}$$

where M is the moment of the external forces with respect to the moving frame of reference. Equations (1.2) and (1.6) are identical in form to the well-known Kirchhoff equations [4] which describe the motion of a rigid body in an ideal fluid.

As an example, consider a special case when O_1 is the centre of mass of the body. Then, $\Sigma m_k\rho_k = 0$ and, hence, $P = mv$, and the quantity (1.5) will be the angular momentum of the body about the centre of mass $I\omega + \lambda$, where

$$I\omega = \sum m_k[\rho_k, [\omega, \rho_k]], \quad \lambda = \sum m_k[\rho_k, \dot{\rho}_k]$$

The symmetric linear operator I is the inertia operator. It is clear that I and λ in general depend on time. In this case, Eq. (1.6) becomes Liouville's equation [1]

$$(I\omega + \lambda)^{\cdot} + [\omega, I\omega + \lambda] = M$$

2. THE EQUATIONS OF MOTION OF A VARIABLE BODY IN A FLUID

We will now assume that the variable body moves in an unbounded volume of an ideal incompressible fluid which undergoes potential flow and is at rest at infinity. To obtain the equations of motion in this case, it is necessary to write the right-hand sides of Eqs (1.2) and (1.6) in explicit form. Suppose S is the boundary of the deformable body. We shall assume that, under the sole action of internal forces, the body is deformed in accordance with a law which is known in advance. Again, we associate the moving Cartesian reference frame $O_1\xi\eta\zeta$ with the body.

Note that the velocity of any point of the body surface S is equal to the sum of the reference-frame and relative velocities. The reference-frame velocity is given by Euler's formula $v + [\omega, \rho]$, where ρ is the radius vector of this point with respect to O_1. The relative velocity is determined by the pure deformation of the body in the moving reference frame. Following [5], we represent the potential of the flow in the form of the sum.

$$\varphi = \sum_{i=1}^{3} v_i\varphi_i + \sum_{i=1}^{3} \omega_i\varphi_{i+3} + \varphi_* \tag{2.1}$$

where v_1, v_2, v_3 (ω_1, ω_2, ω_3) are the components of the velocity vector of the centre of mass (of the angular velocity) in the moving reference frame. The potentials $\varphi_1, \ldots, \varphi_6$ and φ_* are harmonic functions outside the body, which are determined by the impermeability condition: at each point of the boundary of the body $\partial\varphi/\partial n$ (where the outward n is normal to the boundary S) is equal to the normal component of the velocity of this point. The above-mentioned potentials are therefore found as the solutions of the corresponding external Neumann problem. In particular, φ_1, φ_2, φ_3 are the potentials corresponding to the reference-frame motion of the body with unit velocity along the ξ, η, ζ axes and φ_4, φ_5, φ_6 are the potentials corresponding to the rotation of the body at unit angular velocity around these axes. It the boundary of the body S deforms, then (unlike the classical case) the potentials $\varphi_1, \ldots, \varphi_6$ depend on time. The function φ_* is a potential which describes the fluid flow caused by pure deformation of the body.

As is well known (see [6], for example) the kinetic energy of a fluid is found from the formula

$$T_L = -\iint_S \frac{\rho}{2}\varphi\frac{\partial\varphi}{\partial n}d\sigma \tag{2.2}$$

where ρ is the fluid density, $n = (n_1, n_2, n_3)$ is the outward normal to S and $d\sigma$ is an element of area of the surface S. It is clear that $T_c = T_2 + T_1 + T_0$, where T_s is a homogeneous form of degree s in v_i, ω_i. If the boundary of the body is undeformed, then $T_1 = T_0 = 0$ and the coefficients of the quadratic form T_2 (the added masses) are constant. In the general case, the coefficients of the homogeneous forms T_s are time dependent.

It is well-known [6] that the force R and the moment L about the point O_1, acting from the fluid onto the body, have the form

$$R = \frac{d}{dt}\iint_S \rho\varphi n d\sigma, \quad L = \frac{d}{dt}\iint_S \rho\varphi[r, n]d\sigma \tag{2.3}$$

Here r is the radius vector of a point of the surface S with respect to the point O_1. Below we will show that the force R and the moment L can be represented in the following form

$$R = -\frac{d}{dt}\frac{\partial T_L}{\partial v}, \quad L = -\frac{d}{dt}\frac{\partial T_L}{\partial \omega} - \left[v, \frac{\partial T_L}{\partial v}\right] \tag{2.4}$$

This result is well known in the case when there is no pure deformation (see [6], for example). In this case, it is obvious that $T = T_2$.

Substituting formulae (2.4) into Eqs (1.2) and (1.6), we obtain the equation of motion of a variable body in a fluid in the form of Kirchhoff's equations (1.2) and (1.6), where T has the meaning of the total kinetic energy of the "body plus fluid" system.

As an example, we will now prove the first equality of (2.4). To do this, it is sufficient to verify that the vector equality

$$\frac{\partial T}{\partial v} = -\iint_S \rho\varphi n \, d\sigma \tag{2.5}$$

holds.

We will prove that the first components of the vectors (2.5) are equal. Clearly,

$$\frac{\partial T_L}{\partial v_1} = -\frac{\rho}{2}\iint_S \left[\varphi_1 \frac{\partial \varphi}{\partial n} + \varphi \frac{\partial \varphi_1}{\partial n}\right] d\sigma \tag{2.6}$$

Suppose Σ is a sphere of sufficiently large radius with centre at the point O_1, We apply Gauss' formula to the domain V contained between the surfaces S and Σ

$$\iint_\Sigma \left(\varphi_1 \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \varphi_1}{\partial n}\right) d\sigma - \iint_S \left(\varphi_1 \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \varphi_1}{\partial n}\right) d\sigma = \iiint_V (\varphi_1 \Delta\varphi - \varphi\Delta\varphi_1) d\tau \tag{2.7}$$

The right-hand side of this equality is equal to zero since φ and φ_1 are harmonic functions. Suppose a is the radius of the sphere Σ. Then, the integrand in the first integral on the left-hand side of equality (2.7) decreases as $O(a^{-3})$ as $a \to \infty$ (see [5], for example). Hence, the integral over the sphere vanishes when $a \to \infty$. The equality

$$\iint_S \varphi_1 \frac{\partial \varphi}{\partial n} d\sigma = \iint_S \varphi \frac{\partial \varphi_1}{\partial n} d\sigma$$

therefore follows from (2.7).

Taking this equality into account, and also by virtue of the fact that $\partial\varphi_1/\partial n = n_1$ at the points of the surface S, we write equality (2.6) in the form

$$\frac{\partial T_L}{\partial v_1} = -\rho \iint_S \varphi n_1 \, d\sigma$$

Hence, the equality of the first components of the vectors (2.5) has been proved.

3. CONSERVATION LAWS

We will now consider the important special case when there are no external forces acting on a variable body in a fluid: The principal vector of the forces F and principal moment M in Eqs (1.2) and (1.6) are then equal to zero. In this case, we have the two vector integrals, the conservation laws

$$P = \frac{\partial T}{\partial v}, \quad K = \frac{\partial T}{\partial \omega} + \left[r, \frac{\partial T}{\partial v}\right] \tag{3.1}$$

Here, r is the radius vector of the point O_1 with respect to the origin of the fixed reference frame O. Let us show that P and K, as vectors in a fixed space, remain unchanged.

$$\frac{dK}{dt} = \frac{d}{dt}\frac{\partial T}{\partial \omega} + \left[\frac{dr}{dt}, \frac{\partial T}{\partial v}\right] = 0$$

Actually, Eq. (1.6) has the form $dr/dt = v$. The vector P is the total momentum of the "body plus fluid" system and K is the angular momentum of this system about the point O.

It is worth emphasizing that P and K are not only integrals of the dynamic equations (1.2) and (1.6). In order to represent them in explicit form in a moving reference frame, it is necessary to use the matrix of transformation to the moving reference frame. As a result, we obtain six scalar first integrals in v_i and ω_i. On equating these integrals to arbitrary constants, we obtain six independent equations from which it is possible to find v and ω as functions of the position of the moving reference frame. This method, which has been used previously in the conventional Liouville problem [7], enables one to halve the order of the equations of motion of a variable body when there are no external forces. We will now show how this can be done in the important practical special case when the vectors P and K are equal to zero (it can be assumed that the variable body began to move from a state of rest). It has already been mentioned above that the kinetic energy of the "body plus fluid" system has the form

$$T = (Av, v)/2 + (Bv, \omega) + (C\omega, \omega)/2 + (\lambda, v) + (\mu, \omega) + \varkappa \tag{3.2}$$

The matrices A, B and C (A and C are symmetric), the vectors λ and μ and the scalar $\varkappa$ are known functions of time. The homogeneous quadratic form of the kinetic energy with respect to ω and v is positive definite for all values of t. In particular, the symmetric matrices A and C are positive definite and, consequently, non-degenerate. Since $P = 0$ and $K = 0$, we have

$$Av + B^T\omega + \lambda = 0, \qquad Bv + C\omega + \mu = 0$$

Hence,

$$(A - B^T C^{-1} B)v = B^T C^{-1}\mu \quad (C - BA^{-1}B^T)\omega = BA^{-1}\lambda \tag{3.3}$$

Since the quadratic form T_2 is positive definite, the symmetric matrices

$$A - B^T C^{-1} B, \quad C - BA^{-1}B^T$$

are positive (see [8]). Consequently, the vectors ω and v from (3.3) (the angular velocity and the velocity of the centre of mass of the body in the moving reference frame) are found in explicit form as functions of time.

In order to find the motion of the moving reference frame, we introduce the fixed unit vectors α, β, γ directed along the x, y and z axes. Their components in the moving reference frame form an orthogonal transformation matrix. As functions of time, these vectors satisfy Poisson's equations

$$\dot{\alpha} + [\omega, \alpha] = \dot{\beta} + [\omega, \beta] = \dot{\gamma} + [\omega, \gamma] = 0 \tag{3.4}$$

with the already known angular velocity $\omega(t)$. The solutions of the linear system $\alpha(t)$, $\beta(t)$, $\gamma(t)$ with certain initial data uniquely define the orientation of the moving reference frame at the actual instant of time. Finally, the motion of the centre of mass of the body is found by simple integration of the following equations

$$\dot{x} = (v, \alpha), \quad \dot{y} = (v, \beta), \quad \dot{z} = (v, \gamma) \tag{3.5}$$

with known right-hand sides as functions of time.

These remarks are of direct relevance to a problem of the motion of fish: how can a body move in a fluid by means of changing its shape due to the action of internal forces? A model problem of this kind on motion in a solid channel has been considered previously [9] (also, see [10]). It has been shown in [11] that it is possible to generate a tractive force during the motion of a variable body in an ideal vortex-free fluid. The approach is based on the use of the well-known formulae (2.3), but the explicit form of the equations of motion of the body is not given. Relations (3.2)–(3.5) give an algorithm for solving the problem in the most general case. Furthermore, as will be shown in Section 5, a tractive force can be generated without changing the body shape purely by controlling the geometry of its masses.

4. THE PLANE-PARALLEL MOTION OF A VARIABLE BODY

The formulae of Section 3 are greatly simplified in the case of a plane-parallel motion of a body when there are no external forces. Suppose a body moves in such a way that its shape and the distribution of the masses at each instant of time are symmetrical about a certain moving plane Π, x and y are the Cartesian coordinates of the point O_1, that is, the origin of the moving reference frame in the plane Π, and α is the angle of rotation of the moving axes. It is will known that, at each instant of time, the moving reference frame can be chosen in such a way that the kinetic energy (3.2) of the "body plus fluid" system has the form

$$T = (a_1 v_1^2 + a_2 v_2^2 + b\omega^2)/2 + \lambda_1 v_1 + \lambda_2 v_2 + \mu\omega + \varkappa \tag{4.1}$$

Here, $\omega = \dot{\alpha}$ is the angular velocity of the moving reference frame. The coefficients in this formula are unknown functions of time.

Kirchhoff's equations (1.2) and (1.6) take the simpler form

$$\begin{gathered} \left(\frac{\partial T}{\partial v_1}\right)^{\cdot} - \omega\frac{\partial T}{\partial v_2} = 0, \quad \left(\frac{\partial T}{\partial v_2}\right)^{\cdot} + \omega\frac{\partial T}{\partial v_1} = 0 \\ \left(\frac{\partial T}{\partial \omega}\right)^{\cdot} + v_1\frac{\partial T}{\partial v_2} - v_2\frac{\partial T}{\partial v_1} = 0 \end{gathered} \tag{4.2}$$

On adding the simple kinematic relations

$$\dot{x} = v_1\cos\alpha - v_2\sin\alpha, \quad \dot{y} = v_1\sin\alpha + v_2\cos\alpha, \quad \dot{\alpha} = \omega \tag{4.3}$$

to these equations, we obtain a closed system of differential equations, which describes the motion of the moving reference frame.

The first integrals (3.1) of system (4.2), (4.3) have the form

$$\begin{gathered} P_1\frac{\partial T}{\partial v_1}\cos\alpha - \frac{\partial T}{\partial v_2}\sin\alpha, \quad P_2 = \frac{\partial T}{\partial v_1}\sin\alpha + \frac{\partial T}{\partial v_2}\cos\alpha \\ K = xP_2 - yP_1 + \frac{\partial T}{\partial \omega} \end{gathered} \tag{4.4}$$

Hence, taking (4.1) into account,

$$\begin{aligned} v_1 &= (c_1\cos\alpha + c_2\sin\alpha)/a_1 - \lambda_1/a_1 \\ v_2 &= (-c_1\sin\alpha + c_2\cos\alpha)/a_2 - \lambda_1/a_2 \\ \omega &= (c_3 - xc_2 + yc_1)/b - \mu/b \end{aligned}$$

where c_1, c_2, c_3 are the constant values of P_1, P_2, K. On substituting these formulae into the kinematic relations (4.3), we obtain a closed system of differential equations in the group of motions of the plane II. These equations look particularly simple in the case when $c_1 = c_2 = c_3 = 0$ (for example, if the motion, as a result of deformation, starts from a state of rest)

$$\dot{x} = -\frac{\lambda_1\cos\alpha}{a_1} + \frac{\lambda_2\sin\alpha}{a_2}, \quad \dot{y} = -\frac{\lambda_1\sin\alpha}{a_1} - \frac{\lambda_2\cos\alpha}{a_2}, \quad \dot{\alpha} = \frac{\mu}{b} \tag{4.5}$$

We will now discuss the conditions under which periodic changes in the geometry of the masses and the shape of the body lead to non-zero mean values of the velocities $\dot{x}$ and $\dot{y}$. For simplicity, we will use the notation $\xi_k = -\lambda_k/a_k$ $(k = 1, 2)$ and $\eta = -\mu/b$. From (4.5), we obtain the equation for the change in the complex variable $z = x + iy$.

$$\dot{z} = \xi e^{i\alpha}, \quad \xi = \xi_1 + i\xi_2 \tag{4.6}$$

We assume that $\xi(t)$ and $\eta(t)$ are periodic with respect to t with a period $2\pi/\omega$. Since $\dot{\alpha} = \eta$,

$$\alpha(t) = \Omega t + \Phi(t) \tag{4.7}$$

where Ω is the mean value of the function $\eta(t)$ and $\Phi(t)$ is $2\pi/\omega$-periodic in t.

We will now show that, if $\Omega \neq n\omega$ and n is an integer, the mean values of $\dot{x}$ and $\dot{y}$ are zero. In this case, the body will move in a bounded domain. Actually, the equality $\dot{z} = Z(t)\exp i\Omega t$, in which Z is periodic in t with a period $2\pi/\omega$, follows from (4.6) and (4.7). On expanding this function in a Fourier series and integrating with respect to t, we obtain the coordinates of the body

$$z(t) = \sum_{n=-\infty}^{\infty} \frac{Z_n}{i(n\omega + \Omega)} e^{i(n\omega+\Omega)t} + \text{const} \tag{4.8}$$

where Z_n are the Fourier coefficients of the function $Z(t)$. If $\Omega \neq n\omega$, series (4.8) converges and represents a bounded function of t.

The equality $\Omega = 0$ is the simplest resonance relation between the frequencies Ω and ω, at which a tractive force can be created: if $Z_0 \neq 0$ (a typical situation), then the mean value of $\dot{z}$ is non-zero. This situation is of particular interest from the point of view of the problem of the motion of fish. A similar result has previously been obtained by another route in [11] for a model example of the motion of an unbounded body of periodic shape.

5. THE MOTION OF A BODY WITH RIGID BOUNDARY

We will consider the special case of the plane-parallel motion of a body, the shape of the boundary S of which does not change in the moving reference frame. We shall show that, in the case of unequal added masses, a tractive force can be generated by displacements of the points within S under the action of internal forces. Furthermore, with suitable control of the geometry of the masses within S, a body can be displaced from any position to any other position. This effect already manifests itself in the simplest case when just a single point mass is displaced within the material shell.

Thus, we associate a moving system of coordinates $O_1\xi\eta$ with a rigid body such that the kinetic energy of the "body plus fluid" system has the form

$$T' = (a_1 v_1^2 + a_2 v_2^2 + b\omega^2)/2$$

Since the boundary S is not deformed, the coefficients of this form are constant. The motion of a point of mass m is given by certain known functions $\xi(t)$ and $\eta(t)$. The projections of the absolute velocity of this point onto the moving ξ and η axes have the form

$$u_1 = v_1 + \dot{\xi} - \omega\eta, \quad u_2 = v_2 + \dot{\eta} + \omega\xi \tag{5.1}$$

The total kinetic energy of the variable body is equal to

$$T = T' + m(u_1^2 + u_2^2)/2 \tag{5.2}$$

and Kirchhoff's equations have the form (4.2).

We will assume that the body began its motion from a state of rest. Integrals (4.4) then take the form

$$\frac{\partial T}{\partial v_1} = \frac{\partial T}{\partial v_2} = \frac{\partial T}{\partial \omega} = 0$$

Using expression (5.2) and formulae (5.1), we obtain

$$\begin{gathered} v_1 = -\varkappa_1(\dot{\xi} - \omega\eta), \quad v_2 = -\varkappa_2(\dot{\eta} + \omega\xi) \\ \omega = \frac{\varkappa_3[(1-\varkappa_1)\eta\dot{\xi} - (1-\varkappa_2)\xi\dot{\eta}]}{1 + \varkappa_3[(1-\varkappa_1)\eta^2 + (1-\varkappa_2)\xi^2]} \\ \varkappa_1 = \frac{m}{m+a_1}, \quad \varkappa_2 = \frac{m}{m+a_2}, \quad \varkappa_3 = \frac{m}{b} \end{gathered} \tag{5.3}$$

Taking equalities (5.3) into account, the relations

$$\begin{gathered} \dot{x} = X_1(\xi, \eta, \alpha)\dot{\xi} + X_2(\xi, \eta, \alpha)\dot{\eta}, \quad \dot{y} = Y_1(\xi, \eta, \alpha)\dot{\xi} + Y_2(\xi, \eta, \alpha)\dot{\eta} \\ \dot{\alpha} = \Phi_1(\xi, \eta)\dot{\xi} + \Phi_2(\xi, \eta)\dot{\eta} \end{gathered} \tag{5.4}$$

follow from relation (4.3). The explicit form of the coefficients X_k, Y_k, Φ_k ($k = 1, 2$) is easily obtained using formulae (5.3).

It is clear that the position of the rigid body is uniquely specified by the element $z = (x, y, \alpha \bmod 2\pi)$ of the group of motions of the plane. If ξ and η are specified functions of the time t, the position $z(t)$ of the body is found using ordinary quadratures.

It turns out that, if $a_1 \neq a_2$, then, for any $\varepsilon > 0$ and any two positions of the body z_1 and z_2, piecewise-smooth "controlling" functions $\xi(t)$, $\eta(t)$, $t_1 \leqslant t \leqslant t_2$ are found such that

$$|\xi(t)| \leqslant \varepsilon, \quad |\eta(t)| \leqslant \varepsilon; \quad z(t_1) = z_1, \quad z(t_2) = z_2$$

In other words, a body with unequal added masses in a fluid can be displaced from any position to any other position by means of a suitable displacement of a point mass in a specified bounded domain. The time of the motion $t_2 - t_1$ depends very much on ε.

Remark. The condition $a_1 \neq a_2$ is essential. Actually, if $a_1 = a_2 = a$, the two first integrals

$$\begin{aligned} &ax + m(x + \xi\cos\alpha - \eta\sin\alpha) = c_1 \\ &ay + m(y + \xi\sin\alpha + \eta\cos\alpha) = c_2; \qquad c_1, c_2 = \text{const} \end{aligned} \tag{5.5}$$

follow from relations (4.3) and (4.5).

The expressions in parentheses are the coordinates of the point m in the body-fixed axes. If the point O_1 is interpreted as the centre of mass of the "body plus fluid" system, then relation (5.5) denotes the immobility of the centre of mass of the total "body plus fluid plus point mass" system. In particular, it follows from relations (5.5) that the rigid body remains in a bounded domain for the whole of the time (it has to be taken into account that, since the point m remains within the shell of the rigid body, the coordinates ξ and η are bounded).

In order to prove the assertion formulated above, we introduce an extended five-dimensional space M with coordinates x, y, $\alpha \bmod 2\pi$, ξ, η and a distribution of two-dimensional planes defined by the independent equations

$$\begin{aligned} &dx = X_1 d\xi + X_2 d\eta, \qquad dy = Y_1 d\xi + Y_2 d\eta \\ &d\alpha = \Phi_1 d\xi + \Phi_2 d\eta \end{aligned} \tag{5.6}$$

We also introduce the two possible independent vector fields V_1 and V_2 with components X_1, Y_1, Φ_1, 1, 0 and X_2, Y_2, Φ_2, 0, 1 respectively. It is obvious that these vectors satisfy Eqs (5.6). Following the well-known approach [12], we consider the five vector fields

$$V_1, \; V_2, \; [V_1, V_2], \; [V_1, [V_1, V_2]], \; [V_2, [V_1, V_2]] \tag{5.7}$$

where [,] is a Jacobi bracket. If $a_1 \neq a_2$, it is found that, for small values of ξ and η, these vectors are linearly independent at each point.

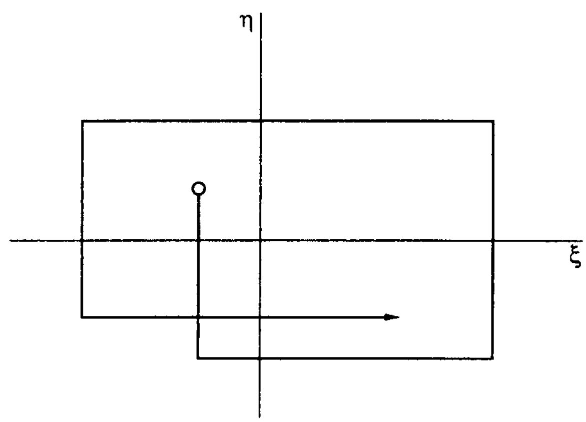

Fig. 2

Actually, the condition of linear independence is equivalent to the fact that the determinant of the 5×5 matrix, consisting of the components of the vectors (5.7), is non-zero. The value of this determinant when $\xi = \eta = 0$ is equal to

$$-(2\varkappa_1 + \varkappa_2 - 3)(2\varkappa_2 + \varkappa_1 - 3)(\varkappa_1 + \varkappa_2 - 2)(\varkappa_1 - \varkappa_2)^2 \varkappa_3^3$$

It is interesting to note that this expression is independent of the angle of rotation α. It is easy to verify that the first three factors are positive. Consequently, if $\varkappa_1 \neq \varkappa_2$ (or, what is the same thing, $a_1 \neq a_2$), the vectors (5.7) are linearly independent for small values of ξ and η.

This fact enables us to use the theorem of Rashevskii [12], according to which any two points of a domain in M, specified by the inequality $|\xi(t)| \leqslant \varepsilon$, $|\eta(t)| \leqslant \varepsilon$ (ε is small), can be joined by a piecewise-smooth curve consisting of segments of the integral curves of the fields V_1 and V_2. It remains to point out that the motion along the phase trajectories of the field V_1 and V_2, parametrized by the time t, satisfies relations (5.4).

Note that the coordinate $\xi(\eta)$ is an integral of the vector field $V_2(V_1)$. Hence, a prespecified motion of a rigid body can be performed by moving the point mass m in the manner qualitatively shown in Fig. 2.

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